Residual symmetry, interaction solutions, and conservation laws of the (2+1)-dimensional dispersive long-wave system *

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We explore the (2+1)-dimensional dispersive long-wave (DLW) system. From the standard truncated Painlevé expansion, the Bäcklund transformation (BT) and residual symmetries of this system are derived. The introduction to an appropriate auxiliary dependent variable successfully localizes the residual symmetries to Lie point symmetries. In particular, it is verified that the (2+1)-dimensional DLW system is consistent Riccati expansion (CRE) solvable. If the special form of (CRE)-consistent tanh-function expansion (CTE) is taken, the soliton-cnoidal wave solutions and corresponding images can be explicitly given. Furthermore, the conservation laws of the DLW system are investigated with symmetries and Ibragimov theorem.

Keywords: residual symmetry, truncated Painlevé expansion, interaction solutions, conservation law

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1. Introduction

It is well known that soliton theory, an important topic in nonlinear science, has been studied thoroughly.[1–5] Many measures can be taken to describe nonlinear phenomena, of which to construct exact solution including the soliton and different wave solutions to an integrable system is an effective one. Symmetry and Painlevé analyses play important roles in deriving the exact solutions.[6–8] Obtained from the classical or nonclassical Lie group method, Lie point symmetries of a differential system can reduce the dimensions of the partial differential equations (PDEs) and help construct group invariant solutions by similarity reductions. Nevertheless, the nonlocal symmetries of the integrable system can be acquired through inverse recursion operators,[9,10] Darboux transformation (DT),[11,12] Bäcklund transformation (BT),[13,14] conformal invariance,[15] negative hierarchies,[16] and so on.

Recently, Lou and his colleagues[17,18] found that Painlevé analysis can also be applicable in acquiring nonlocal symmetries, also known as residual symmetries, since the symmetries correspond to the residues with respect to the singular manifold of the truncated Painlevé expansion. However, the nonlocal symmetries cannot be directly used to establish explicit solutions to differential equations, so the transformation of the nonlocal symmetries into local ones is needed with the help of suitable prolonged systems.[18,19] In Ref. [18], the residual symmetries are localized and the related finite transformation was found by Lou. Furthermore, advancing the truncated Painlevé expansion, Lou introduced the definition of consistent Riccati expansion (CRE) solvable,[20] which is of great efficiency in constructing interaction solutions and possible new integrable systems. In Ref. [21], a consistent tanh expansion (CTE) was proposed to identify CTE solvable, a special simplified form of the CRE.

On the other hand, conservation laws, essential in the study of differential equations mathematically and physically, propose one of the primary principles to formulate and investigate models, especially in existence, uniqueness, and stability of the solutions. In addition, the integrability of the system is quite possible if conservation laws exist.[22] For conservation laws, different methods have been mobilized. The celebrated Noether's theorem[23] proves to be a systematic and efficient approach in finding conservation laws of PDEs unless there exists a Lagrangian. However, there exist some equations not having a Lagrangian. Hence the Noether theorem cannot be used to obtain conservation laws directly because of the equation symmetries. This, however, can be solved with the general concept of nonlinear self-adjointness proposed by Ibragimov[24,25] and Gandarias[26] to construct the conservation laws for any differential equation. This procedure can be true of classes of single differential equations of any order but of the systems where the number of equations is equal to that of dependent variables.[27]

This paper concentrates on investigating the residual symmetries, CRE solvable, interaction solution, and conservation laws of the (2+1)-dimensional dispersive long-wave (DLW)

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system

$$u_{yt}+v_{xx}+u_xu_y+uu_{xy}=0,\quad v_t+(uv+u+u_{xy})_x=0. \quad (1)$$

System (1), modeling nonlinear and dispersive long gravity waves in two horizontal directions on shallow waters of uniform depth, was initially proposed by Boiti *et al.*[28] By now, system (1) has been extensively studied by many professionals. Reference [29] has presented some new soliton-like solutions to system (1). Reference [30] provided the rational series solutions and multi solitary wave solutions to system (1), while in Refs. [31] and [32], with the extended mapping method and variable separation approach, respectively, new variable separation excitations with arbitrary function and numerous localized coherent structures such as multi-ring solitons, dromion, breathers, and instantons of system (1) have been obtained. In Ref. [33], the generalized singular manifold approach yields the Darboux transformations and Lax pair of systems (1). In Ref. [34], two kinds of new multiple soliton solutions have been constructed through the Bäcklund transformation and Hirota bilinear method, and the fusion and fission interaction phenomena among the different localized structures have been used to investigate into system (1). However, we can see that the residual symmetry, soliton-cnodial wave solution, and conservation laws of system (1) have not been studied yet in the above literature.

This paper is organized as follows. Section 2 focuses on the non-auto Bäcklund transformation theorem and residual symmetry of the (2+1)-dimensional DLW system by using the truncated Painlevé expansion approach. Then extending the original system makes the residual symmetry localized. Consequently, the corresponding finite transformation group can be obtained with Lie's first theorem. Section 3 verifies that the (2+1)-dimensional DLW system is CRE solvable and leads to new interactions between solitons and cnoidal periodic waves. In Section 4, the conservation laws of the (2+1)-dimensional DLW system are derived from the Ibragimov theorem. In the last section, some conclusions and a discussion are presented.

2. The residual symmetry of DLW system and its localization

For the (2+1)-dimensional DLW system, its truncated Painlevé expansion can be expressed as

$$u=\frac{u_1}{\phi}+u_0,\quad v=\frac{v_2}{\phi^2}+\frac{v_1}{\phi}+v_0, \quad (2)$$

with u_0, u_1, v_0, v_1, v_2, ϕ being the functions of x, y, and t. Substituting Eq. (2) into system (1) and eliminating all the coefficients of different powers of $1/\phi$, we have

$$u_0=-\frac{\phi_{xx}+\phi_t}{\phi_x},\quad u_1=2\phi_x,$$

$$\begin{aligned}&v_1=2\phi_{xy},\quad v_2=-2\phi_x\phi_y,\\ &v_0=-\frac{\phi_x\phi_{xxy}-\phi_{xy}\phi_{xx}+\phi_x^2+\phi_x\phi_{yt}-\phi_{xy}\phi_t}{\phi_x^2};\end{aligned} \quad (3)$$

and the field ϕ satisfies the following Schwarzian form:

$$P_t+S_x+2P_{xx}-PP_x-k=0, \quad (4)$$

where k is an arbitrary integral parameter,

$$P=\frac{\phi_t}{\phi_x},\quad S=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2.$$

The Schwarzian form (4) is invariant under the Möbious transformation

$$\phi\to\frac{a+b\phi}{c+d\phi}\quad(ad\neq bc).$$

That is to say, equation (4) bears three symmetries $\sigma^\phi=d_1$, $\sigma^\phi=d_2\phi$, and $\sigma^\phi=d_3\phi^2$ with arbitrary constants d_1, d_2, and d_3.

After substituting expression (2) into system (1), we have the following theorem.

Theorem 1 (non-auto BT theorem) If the field ϕ meets Eq. (4), then

$$\begin{aligned}&u=-\frac{\phi_{xx}+\phi_t}{\phi_x},\\ &v=-\frac{\phi_x\phi_{xxy}-\phi_{xy}\phi_{xx}+\phi_x^2+\phi_x\phi_{yt}-\phi_{xy}\phi_t}{\phi_x^2}\end{aligned} \quad (5)$$

is a non-auto BT between ϕ and the solution u,v of the (2+1)-dimensional DLW system (1).

Theorem 2 The DLW system (1) has the following residual symmetry:

$$\sigma^u=2\phi_x,\quad \sigma^v=2\phi_{xy}, \quad (6)$$

where u, v, and ϕ satisfy the non-auto BT (5).

Proof The symmetry equations of system (1) are

$$\begin{aligned}&\sigma^u_{yt}+\sigma^v_{xx}+\sigma^u_xu_y+u_x\sigma^u_y+\sigma^uu_{xy}+u\sigma^u_{xy}=0,\\ &\sigma^v_t+v\sigma^u_x+\sigma^vu_x+\sigma^uv_x+u\sigma^v_x+\sigma^u_x+\sigma^u_{xxy}=0.\end{aligned} \quad (7)$$

Equations (7) hold when substituting Eq. (6) into Eq. (7) with the help of Eqs. (4) and (5).

Since nonlocal symmetries cannot be used to construct explicit solutions to differential equations directly, nonlocal symmetries need to be transformed into local ones. To this end, we introduce new variables f, g, and f_1 to eliminate the space derivatives of ϕ by

$$f=\phi_x,\quad g=\phi_y,\quad f_1=f_y. \quad (8)$$

Now the nonlocal symmetry (6) of the DLW system (1) is localized to a Lie point symmetry

$$\begin{aligned}&\sigma^u=2f,\quad \sigma^v=2f_1,\quad \sigma^\phi=-\phi^2,\quad \sigma^f=-2\phi f,\\ &\sigma^g=-2\phi g,\quad \sigma^{f_1}=-2(gf+\phi f_1).\end{aligned}$$

for the prolonged equations (1), (4), and (8) with the Lie point symmetry vector

$$\begin{aligned} V &= 2f\partial_u + 2f_1\partial_v - \phi^2\partial_\phi - 2\phi f\partial_f \\ &\quad - 2\phi g\partial_g - 2(gf+\phi f_1)\partial_{f_1}. \end{aligned} \tag{9}$$

To proceed, we study the finite symmetry transformation of Lie point symmetry (9). According to Lie's first theorem, solving the initial value problem

$$\begin{aligned} &\frac{\mathrm{d}\hat{u}(\varepsilon)}{\mathrm{d}\varepsilon} = 2\hat{f}(\varepsilon), \quad \hat{u}(0)=u, \quad \frac{\mathrm{d}\hat{v}(\varepsilon)}{\mathrm{d}\varepsilon} = 2\hat{f}_1(\varepsilon), \quad \hat{v}(0)=v, \\ &\frac{\mathrm{d}\hat{\phi}(\varepsilon)}{\mathrm{d}\varepsilon} = -\hat{\phi}^2(\varepsilon), \quad \hat{\phi}(0)=\phi, \\ &\frac{\mathrm{d}\hat{f}(\varepsilon)}{\mathrm{d}\varepsilon} = -2\hat{f}(\varepsilon)\hat{\phi}(\varepsilon), \quad \hat{f}(0)=f, \\ &\frac{\mathrm{d}\hat{f}_1(\varepsilon)}{\mathrm{d}\varepsilon} = -2(\hat{g}(\varepsilon)\hat{f}(\varepsilon)+\hat{\phi}(\varepsilon)\hat{f}_1(\varepsilon), \quad \hat{f}_1(0)=f_1, \\ &\frac{\mathrm{d}\hat{g}(\varepsilon)}{\mathrm{d}\varepsilon} = -2\hat{g}(\varepsilon)\hat{\phi}(\varepsilon), \quad \hat{g}(0)=g \end{aligned} \tag{10}$$

will easily yield the following symmetry group transformation theorem.

Theorem 3 If $\{u,v,\phi,f,g,f_1\}$ is a solution to the prolonged system (1), (4), and (8), $\{\hat{u},\hat{v},\hat{\phi},\hat{f},\hat{g},\hat{f}_1\}$ is

$$\begin{aligned} &\hat{u} = -\frac{2f}{(\varepsilon\phi+1)\phi} + \frac{u\phi+2f}{\phi}, \\ &\hat{v} = v + \frac{2\varepsilon^2(\phi f_1 - fg)+2\varepsilon f_1}{(\phi\varepsilon+1)^2}, \quad \hat{\phi} = \frac{\phi}{\varepsilon\phi+1}, \\ &\hat{f} = \frac{f}{(\varepsilon\phi+1)^2}, \quad \hat{g} = \frac{g}{(\varepsilon\phi+1)^2}, \\ &\hat{f}_1 = \frac{f_1(\varepsilon\phi+1)-2fg\varepsilon}{(\varepsilon\phi+1)^3}. \end{aligned} \tag{11}$$

Remark 1 In Theorem 3, we find the residual symmetry $\sigma^u = 2\phi_x, \sigma^v = 2\phi_{xy}$ resulted from the truncated Painlevé expansion is just the infinitesimal form of the group. In fact, the above group transformation is equivalent to the truncated Painlevé expansion (2) and (3) because the singularity manifold equations (1), (4), and (8) are form invariant under the transformation

$$1+\varepsilon\phi \to \phi \ \text{ with } \ (\varepsilon f \to \phi_x, \varepsilon g \to \phi_y, \varepsilon f_1 \to \phi_{xy}).$$

3. CRE solvability and interaction solutions to DLW system

3.1. CRE solvability

The leading order analysis lays the basis for the acquisition of the following truncated Painlevé expansion solution to the DLW system:

$$\begin{aligned} &u = u_0 + u_1 R(w), \\ &v = v_0 + v_1 R(w) + v_2 R^2(w), \quad w = w(x,y,t), \end{aligned} \tag{12}$$

in which $R(w)$ is a solution to the Riccati equation

$$R_w = a_0 + a_1 R + a_2 R^2, \tag{13}$$

with a_0, a_1, and a_2 being arbitrary constants. Substituting Eqs. (12) and (13) into Eq. (1) and then vanishing all the coefficients of the like powers of $R(w)$, we obtain

$$\begin{aligned} u_0 &= \frac{a_1 w_x^2 + w_{xx} - w_t}{w_x}, \\ v_0 &= (2a_0a_2w_x^3w_y + a_1w_{xy}w_x^2 + w_xw_{xxy} - w_{xx}w_{xy} \\ &\quad + w_{xy}w_t + w_x^2 - w_xw_{yt})/-w_x^2, \\ u_1 &= 2a_2w_x, \quad v_1 = -2a_2(a_1w_xw_y + w_{xy}), \\ v_2 &= -2a_2^2 w_x w_y, \end{aligned} \tag{14}$$

where the function w only needs to satisfy

$$\begin{aligned} &P_{1t} + S_{1x} - 2P_{1xx} - P_1P_{1x} - \delta w_x w_{xx} = 0, \\ &(\delta = a_1^2 - 4a_0a_2), \end{aligned} \tag{15}$$

with $P_1 = \frac{w_t}{w_x}$ and $S_1 = \frac{w_{xxx}}{w_x} - \frac{3}{2}(\frac{w_{xx}}{w_x})^2$.

From the above, it can be concluded that the (2+1)-dimensional DLW system is CRE solvable since it has the truncated Painlevé expansion solution related to the Riccati equation (13), and the following theorem can be established.

Theorem 4 The DLW system (1) is CRE solvable with the CRE

$$\begin{aligned} u &= u_0 + 2a_2 w_x R(w), \\ v &= v_0 - 2a_2(a_1w_xw_y + w_{xy})R(w) - 2a_2^2 w_x w_y R^2(w), \end{aligned} \tag{16}$$

where u_0 and v_0 are determined by Eq. (14).

3.2. CTE solvable and interaction solutions to the DLW system

It is known that the Riccati equation (13) has a special solution $R(w) = \tanh(w)$ when $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, which substantiates that the CRE solvable system is CTE solvable, and vice versa. Then the following CTE solvable theorem can be acquired.

Theorem 5 The DLW system (1) is CTE solvable with the CTE

$$\begin{aligned} u &= u_0 + 2a_2 w_x \tanh(w), \\ v &= v_0 - 2a_2(a_1w_xw_y + w_{xy})\tanh(w) \\ &\quad - 2a_2^2 w_x w_y \tanh^2(w), \end{aligned} \tag{17}$$

where u_0 and v_0 are determined by Eq. (14) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$.

Theorem 5 shows that solving the w equation (15) can result in various interaction solutions between solitons and other nonlinear excitations, which posess a form[36]

$$w = h_1x + h_2y + h_3t + W(q_1x + q_2y + q_3t), \tag{18}$$

where $W(q_1x+q_2y+q_3t)=W(X)=W$, and $W_1(X)=W_X$ satisfies

$$W_{1X}^2 = b_0+b_1W_1(X)+b_2W_1(X)^2 +b_3W_1(X)^3+b_4W_1(X)^4, \tag{19}$$

with b_0, b_1, b_2, b_3, b_4 being constants. Substituting Eqs. (18) and (19) into Eq. (15), with the help of software Maple, we obtain

$$b_0=\frac{h_1^2(b_2q_1^2-2b_3h_1q_1+9\delta h_1^2)}{q_1^4},$$
$$b_1=\frac{h_1(2b_2q_1^2-3b_3h_1q_1+12\delta h_1^2)}{q_1^3},$$
$$b_4=3\delta,\ \ h_3=\frac{h_1q_3}{q_1}, \tag{20}$$

while all the other constants h_1, h_2, h_3, q_1, q_2, q_3, b_2, and b_3 remain free.

Obviously, the general solution to Eq. (19) can be expressed in terms of Jacobi elliptic functions. Here, just take w in the following special Jacobi elliptic function

$$w = h_1x+h_2y+h_3t+\lambda E_\pi \times(\mathrm{sn}(q_1x+q_2y+q_3t,m),n,m) \tag{21}$$

as the solution to Eq. (15), which characterizes the interactions between a soliton and a cnoidal wave. Here, $\mathrm{sn}(z,m)$ is the usual Jacobi elliptic sine function and

$$E_\pi(\zeta,n,m)=\int_0^\zeta \frac{\mathrm{d}t}{(1-nt^2)\sqrt{(1-t^2)(1-m^2t^2)}} \tag{22}$$

is the third type of incomplete elliptic integral. Substituting Eq. (21) into Eq. (15) and solving the over-determined equations with the help of maple will yield

$$\{h_1=h_1,\ h_2=h_2,\ h_3=h_3,\ \lambda=\lambda,\ m=m,\ n=0,\ q_1=q_1,\ q_2=q_2,\ q_3=q_3\}, \tag{23}$$
$$\{h_1=2Zq_1,\ h_2=h_2,\ h_3=2Zq_3,\ \lambda=-4Z,\ m=-1,\ n=-1,\ q_1=q_1,\ q_2=q_2,\ q_3=q_3\}, \tag{24}$$
$$\{h_1=2Zq_1,\ h_2=h_2,\ h_3=2Zq_3,\ \lambda=-4Z,\ m=1,\ n=-1, q_1=q_1,\ q_2=q_2,\ q_3=q_3\}, \tag{25}$$

where

$$Z=\frac{1}{\sqrt{(4a_0a_2-a_1^2)}} \quad \text{or} \quad -\frac{1}{\sqrt{(4a_0a_2-a_1^2)}}, \tag{26}$$

where $(4a_0a_2-a_1^2)\neq 0$, h_1, h_2, h_3, λ, q_1, q_2, q_3, and m in Eq. (23) are arbitrary constants, and $h_3=h_1q_3/q_1$.

When equation (23) is substituted into expression (21), then equation (21) will become

$$w = h_1x+h_2y+h_3t+\lambda \mathrm{EllipticF}(\mathrm{JacobiSN} \times(q_1x+q_2y+q_3t,m),m) \tag{27}$$

where

$$\mathrm{EllipticF}(z,k)=\int_0^z \frac{\mathrm{d}\alpha}{\sqrt{(1-\alpha^2)}\sqrt{(1-\alpha^2k^2)}}$$

is the first type of incomplete elliptic integral. Substituting Eqs. (14) and (27) into Eq. (17), we can obtain the interaction solution between soliton and cnoidal periodic waves. The result is omitted here because of its prolixity. The corresponding images are shown in Figs. 1 and 2, and the parameters used in the figure are selected as $\{h_1=1,\ h_2=0.5,\ h_3=0.5,\ \lambda=0.1,\ q_1=2,\ q_2=4,\ q_3=1,\ m=0.5\}$.

When equation (24) or (25) with $a_0=1$, $a_1=0$, and $a_2=-1$ is substituted into expression (21), then equation (21) becomes

$$w = -\mathrm{i}(\mathrm{i}h_2y+q_1x+q_3t-2\mathrm{EllipticF} \times \tanh(q_1x+q_2y+q_3t,-1),1)). \tag{28}$$

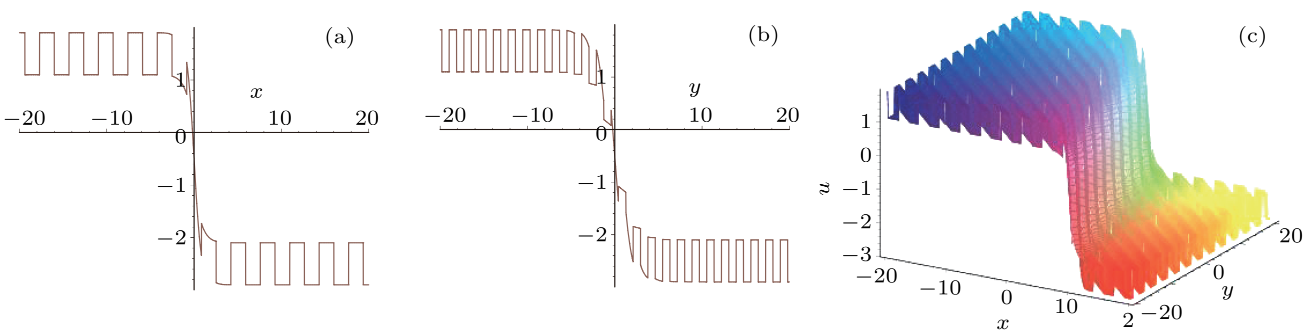

Fig. 1. (color online) The soliton-cnodial periodic wave solutions to u: (a) the profile of the special structure with $t=0$ and $y=0$, (b) the profile of the special structure with $t=0$ and $x=0$, (c) perspective view of the wave.

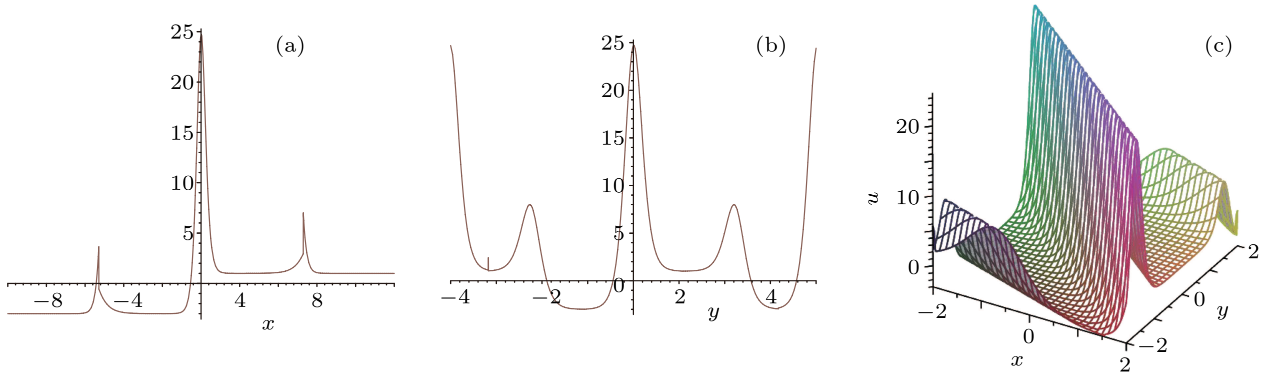

Fig. 2. (color online) The soliton-cnodial periodic wave solutions to v: (a) the profile of the special structure with $t = 0$ and $y = 0$, (b) the profile of the special structure at $t = 0$ and $x = 0$, (c) perspective view of the wave.

Substituting Eqs. (14) and (28) into Eq. (17) will lead to the following two-soliton solutions of system (1):

$$
\begin{aligned}
u = & \frac{1}{(\tanh(q_1x+q_2y+q_3t)^2+1)q_1}(2Tq_1^2-q_3) \\
& \times \tanh(q_1x+q_2y+q_3t)^2 \\
& -4q_1^2\tanh(q_1x+q_2y+q_3t)-2q_1^2T-q_3, \qquad (29)\\
v = & -\frac{1}{(\tanh(q_1x+q_2y+q_3t)^2+1)^2} \\
& \times((2\mathrm{i}T^2h_2q_1+2\mathrm{i}h_2q_1-4q_1q_2+1) \\
& \times \tanh(q_1x+q_2y+q_3t)^4-8Tq_1q_2\tanh(q_1x+q_2y+q_3t)^3 \\
& +2(1+2q_1q_2-2T^2q_1q_2)\tanh(q_1x+q_2y+q_3t)^2 \\
& +8Tq_1q_2\tanh(q_1x+q_2y+q_3t) \\
& -2\mathrm{i}T^2h_2q_1+4T^2q_1q_2-2\mathrm{i}h_2q_1+1), \qquad (30)
\end{aligned}
$$

where $T = \tan(\mathrm{i}h_2y+q_1x+q_3t-2\mathrm{EllipticF}(\tanh(q_1x+q_2y+q_3t,-1),1))$.

Since the solutions are complex numbers, the figures drawn here are modules to them (Fig. 3), and the parameters used in the figure are selected as $\{t = 0,\ h_2 = 0.1,\ h_3 = 0.5,\ q_1 = 0.3,\ q_2 = 0.3,\ q_3 = 0.2\}$.

Remark 2 Figures 1 and 2 illustrate the soliton-cnoidal periodic wave solutions to the fields u and v, describing a soliton that travels on a cnoidal wave background for the DLW system. Clearly, the interaction between the soliton and every peak of the cnoidal periodic wave is elastic as phase changes. Solutions and figures obtained in this paper might be helpful in further understanding the propagation of nonlinear and dispersive long gravity waves on shallow waters. When the module $m = 1$ or $m = -1$ and $n = -1$, the soliton-cnoidal periodic wave solutions will reduce back to the two-soliton solutions as shown in Fig. 3. Figure 3(b) shows that two line solitons intersecting in the initial time ($t = 0$) produce a resonance phenomenon.

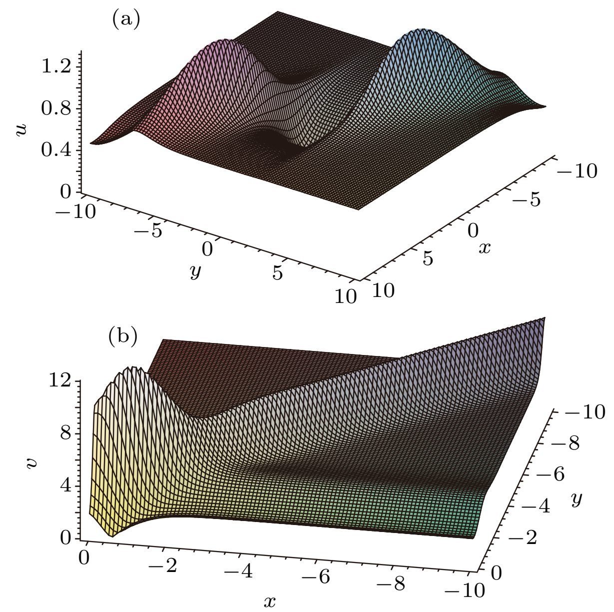

Fig. 3. (color online) The two soliton-solution structure of (a) u and (b) v determined by Eqs. (29) and (30) for the DLW system at t=0.

4. Conservation laws for DLW system

In this section, we consider the conservation laws of the DLW system (1) by Ibragimov's theorem with the introduction of some notations and theorems.

Definition 1[24] Consider a system of equations

$$F_\alpha(x,u,u_{(1)},\ldots,u_{(s)}) = 0, \quad \alpha = 1,\ldots,m, \qquad (31)$$

with n independent variables $x = (x^1,\ldots,x^n)$ and m dependent variables $u = (u^1,\ldots,u^m)$ where $u_{(s)}$ denotes the set of the partial derivatives of s-th order of u. The adjoint equation to Eq. (31) is

$$F^*_\alpha(x,u,v,\ldots,u_{(s)},v_{(s)}) = 0, \quad \alpha = 1,\ldots,m, \qquad (32)$$

with

$$F^*_\alpha(x,u,v,\ldots,u_{(s)},v_{(s)}) = \frac{\delta L}{\delta u^\alpha}, \quad \alpha = 1,\ldots,m, \qquad (33)$$

where L is the formal Lagrangian for Eq. (31) given by

$$L = \sum_{\beta=1}^{m} v^\beta F_\beta(x,u,u_{(1)},\ldots,u_{(s)}), \qquad (34)$$

$v=(v^1,\ldots,v^m)$ are new dependent variables, $v=v(x)$, and $\delta/\delta u^\alpha$ is the variational derivative

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s=1}^{\infty}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots s}}.$$

Theorem 6[24] Any infinitesimal symmetry (Lie point, Lie Bäcklund, nonlocal)

$$X=\xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u^\alpha}$$

of Eq. (31) provides a conservation law $D_i(C^i)=0$ for the system of differential equations consisting of Eq. (31) and the adjoint equations (32). The conserved vector is given by

$$\begin{aligned}C^i&=\xi^iL+W^\alpha\left[\frac{\partial L}{\partial u^\alpha_i}-D_j\left(\frac{\partial L}{\partial u^\alpha_{ij}}\right)+D_jD_k\left(\frac{\partial L}{\partial u^\alpha_{ijk}}\right)-\cdots\right]\\&\quad+D_j(W^\alpha)\left[\frac{\partial L}{\partial u^\alpha_{ij}}-D_k\left(\frac{\partial L}{\partial u^\alpha_{ijk}}\right)-\cdots\right]\\&\quad+D_jD_k(W^\alpha)\left[\frac{\partial L}{\partial u^\alpha_{ijk}}-\cdots\right],\end{aligned}\tag{35}$$

and $W^\alpha=\eta^\alpha-\xi^ju^\alpha_j$.

We utilize Theorem 6 to construct the conservation law for system (1). For system (1), we introduce the Lagrangian in the following symmetrized form

$$\begin{aligned}L&=u_1^*\left(\frac12u_{yt}+\frac12u_{ty}+v_{xx}+u_xu_y+u\left(\frac12u_{xy}+\frac12u_{yx}\right)\right)\\&\quad+v_1^*\left(v_t+u_xv+uv_x+u_x+\frac13u_{xxy}\right.\\&\quad\left.+\frac13u_{xyx}+\frac13u_{yxx}\right),\end{aligned}\tag{36}$$

where u_1^*, v_1^* are new dependent variables, and the adjoint equations of system (1) from definition 1 are

$$\begin{aligned}&u^*_{1yt}+uu^*_{1xy}-vv^*_{1x}-v^*_{1x}-v^*_{1xxy}=0,\\&v^*_{1t}+uv^*_{1x}-u^*_{1xx}=0\end{aligned}\tag{37}$$

Based on Theorem 6, for a general vector field

$$X=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v},\tag{38}$$

the conservation law is under the control of

$$D_t(C^1)+D_x(C^2)+D_y(C^3)=0.\tag{39}$$

Here the conserved vector $\boldsymbol{C}=(C^1,C^2,C^3)$ of Eq. (35) will change into the following concrete forms

$$\begin{aligned}C^1&=\xi^1L+W^2\frac{\partial L}{\partial v_t}-W^1D_y\frac{\partial L}{\partial u_{ty}}+W^1_y\frac{\partial L}{\partial u_{ty}},\\C^2&=\xi^2L+W^1\left(\frac{\partial L}{\partial u_x}-D_y\frac{\partial L}{\partial u_{xy}}+D_{xy}\frac{\partial L}{\partial u_{xxy}}\right.\\&\quad\left.+D_{yx}\frac{\partial L}{\partial u_{xyx}}\right)+W^2\left(\frac{\partial L}{\partial v_x}-D_x\frac{\partial L}{\partial v_{xx}}\right)\\&\quad+W^1_y\left(\frac{\partial L}{\partial u_{xy}}-D_x\frac{\partial L}{\partial u_{xyx}}\right)-W^1_xD_y\frac{\partial L}{\partial u_{xxy}}\\&\quad+W^2_x\frac{\partial L}{\partial v_{xx}}+W^1_{xy}\frac{\partial L}{\partial u_{xxy}}+W^1_{yx}\frac{\partial L}{\partial u_{xyx}},\\C^3&=\xi^3L+W^1\left(\frac{\partial L}{\partial u_y}-D_x\frac{\partial L}{\partial u_{yx}}-D_t\frac{\partial L}{\partial u_{yt}}\right.\\&\quad\left.+D_{xx}\frac{\partial L}{\partial u_{yxx}}\right)+W^1_t\frac{\partial L}{\partial u_{yt}}\\&\quad+W^1_x\left(\frac{\partial L}{\partial u_{yx}}-D_x\frac{\partial L}{\partial u_{yxx}}\right)+W^1_{xx}\frac{\partial L}{\partial u_{yxx}}.\end{aligned}\tag{40}$$

By substituting Eq. (36) into Eq. (40), it will change into

$$\begin{aligned}C^1&=\xi^1L+W^2v_1^*-\frac12(W^1u^*_{1y}-W^1_yu^*_1),\\C^2&=\xi^2L+W^1\left(\frac12u^*_1u_y+v^*_1v-\frac12uu^*_{1y}+\frac23v^*_{1xy}\right)\\&\quad+W^2(v^*_1u-u^*_{1x})+W^1_y\left(\frac12uu^*_1-\frac13v^*_{1x}\right)\\&\quad-\frac13W^1_xv^*_{1y}+W^2_xu^*_1+\frac23W^1_{xy}v^*_1,\\C^3&=\xi^3L+W^1\left(\frac12u^*_1u_x-\frac12u^*_{1t}-\frac12u^*_{1x}u+\frac13v^*_{1xx}\right)\\&\quad+W^1_x\left(\frac12uu^*_1-\frac13v^*_{1x}\right)+\frac13W^1_{xx}v^*_1+\frac12W^1_tu^*_1,\end{aligned}$$

with

$$\begin{aligned}W^1&=\eta^1-\xi^1u_t-\xi^2u_x-\xi^3u_y,\\W^2&=\eta^2-\xi^1v_t-\xi^2v_x-\xi^3v_y.\end{aligned}$$

The following symmetries of system (1) are obtained from the classical Lie group theory[35]

$$\begin{aligned}X_1&=-\frac12xf_{1t}\frac{\partial}{\partial x}+f_1\frac{\partial}{\partial t}-\frac12(uf_{1t}+xf_{1tt})\frac{\partial}{\partial u}\\&\quad+\frac12f_{1t}(v+1)\frac{\partial}{\partial v},\end{aligned}\tag{41}$$

$$X_2=f_2\frac{\partial}{\partial y}+f_{2y}(v+1)\frac{\partial}{\partial v},\tag{42}$$

$$X_3=f_3\frac{\partial}{\partial x}+f_{3t}\frac{\partial}{\partial u},\tag{43}$$

where f_2 is an arbitrary function of y, f_1,f_3 of t.

To proceed, we consider the following cases.

Case 1 For the generator

$$\begin{aligned}X_1&=-\frac12xf_{1t}\frac{\partial}{\partial x}+f_1\frac{\partial}{\partial t}-\frac12(uf_{1t}+xf_{1tt})\frac{\partial}{\partial u}\\&\quad+\frac12f_{1t}(v+1)\frac{\partial}{\partial v},\end{aligned}$$

the Lie characteristic functions are

$$\begin{aligned}W^1&=-\frac12(uf_{1t}+xf_{1tt})+\frac12xf_{1t}u_t-f_1u_x,\\W^2&=\frac12f_{1t}(v+1)+\frac12xf_{1t}v_t-f_1v_x,\end{aligned}$$

one can obtain the conservation vector of system (1)

$$
\begin{aligned}
C^1 =& -\frac{1}{2}xf_{1t}[u_1^*(u_{yt}+v_{xx}+u_xu_y+uu_{xy})\\
&+v_1^*(v_t+u_xv+uv_x+u_x+u_{xxy})]\\
&+\left[\frac{1}{2}f_{1t}(v+1)+\frac{1}{2}xf_{1t}v_t-f_1v_x\right]v_1^*\\
&+\frac{1}{4}(uf_{1t}+xf_{1tt}-xf_{1t}u_t)u_{1y}^*\\
&+\frac{1}{2}f_1u_xu_{1y}^*+\frac{1}{2}u_1^*\left(\frac{1}{2}xf_{1t}u_{ty}-f_1u_{xy}-\frac{1}{2}u_yf_{1t}\right),\\
C^2 =& f_1[u_1^*(u_{yt}+v_{xx}+u_xu_y+uu_{xy})\\
&+v_1^*(v_t+u_xv+uv_x+u_x+u_{xxy})]\\
&-\left[\frac{1}{2}(uf_{1t}+xf_{1tt})-\frac{1}{2}xf_{1t}u_t+f_1u_x\right]\\
&\times\left(\frac{1}{2}u_1^*u_y+v_1^*v-\frac{1}{2}uu_{1y}^*+\frac{2}{3}v_{1xy}^*\right)\\
&+\left[\frac{1}{2}f_{1t}(v+1)+\frac{1}{2}xf_{1t}v_t-f_1v_x\right](v_1^*u-u_{1x}^*)\\
&+\left(\frac{1}{2}xf_{1t}u_{ty}-f_1u_{xy}-\frac{1}{2}f_{1t}u_y\right)\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)\\
&-\frac{1}{3}\left[\frac{1}{2}f_{1t}u_t+\frac{1}{2}xf_{1t}u_{tx}-f_1u_{xx}-\frac{1}{2}(u_xf_{1t}+f_{1tt})\right]v_{1y}^*\\
&+\left(\frac{1}{2}f_{1t}v_x+\frac{1}{2}f_{1t}v_t+\frac{1}{2}xf_{1t}v_{tx}-f_1v_{xx}\right)u_1^*\\
&+\frac{2}{3}\left(\frac{1}{2}f_{1t}u_{ty}+\frac{1}{2}xf_{1t}u_{txy}-f_1u_{xxy}-\frac{1}{2}u_{xy}f_{1t}\right)v_1^*,\\
C^3 =& \left[\frac{1}{2}xf_{1t}u_t-\frac{1}{2}(uf_{1t}+xf_{1tt})-f_1u_x\right]\\
&\times\left(\frac{1}{2}u_1^*u_x-\frac{1}{2}u_{1t}^*-\frac{1}{2}uu_{1x}^*+\frac{1}{3}v_{1xx}^*\right)\\
&+\left[\frac{1}{2}f_{1t}u_t+\frac{1}{2}xf_{1t}u_{tx}-f_1u_{xx}-\frac{1}{2}(u_xf_{1t}+f_{1tt})\right]\\
&\times\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)+\frac{1}{3}\left(f_{1t}u_{tx}+\frac{1}{2}xf_{1t}u_{txx}\right.\\
&\left.-f_1u_{xxx}-\frac{1}{2}f_{1t}u_{xx}\right)v_1^*\\
&+\frac{1}{2}\left[\frac{1}{2}x(f_{1tt}u_t+f_{1t}u_{tt})-f_{1t}u_x-f_1u_{xt}\right.\\
&\left.-\frac{1}{2}(f_{1t}u_t+f_{1tt}u+xf_{1ttt})\right]u_1^*.
\end{aligned}
$$

Case 2 For the generator

$$X_2=f_2\frac{\partial}{\partial y}+f_{2y}(v+1)\frac{\partial}{\partial v},$$

the Lie characteristic functions are

$$W^1=-f_2u_y,\ \ W^2=-f_{2y}(v+1)-f_2v_y,$$

we can obtain the conservation vector of system (1)

$$
\begin{aligned}
C^1 =& -[f_{2y}(v+1)+f_2v_y]v_1^*+\frac{1}{2}f_2u_yu_{1y}^*\\
&-\frac{1}{2}(f_{2y}u_y+f_2u_{yy})u_1^*,\\
C^2 =& -f_2u_y\left(\frac{1}{2}u_1^*u_y+v_1^*v-\frac{1}{2}uu_{1y}^*+\frac{2}{3}v_{1xy}^*\right)\\
&-[f_{2y}(v+1)+f_2v_y](v_1^*u-u_{1x}^*)\\
&-(f_{2y}u_y+f_2u_{yy})\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)\\
&+\frac{1}{3}f_2u_{xy}v_{1y}^*-(f_{2y}v_x+f_2v_{xy})u_1^*\\
&-\frac{2}{3}(f_{2y}u_{xy}+f_2u_{xyy})v_1^*,\\
C^3 =& f_2[u_1^*(u_{yt}+v_{xx}+u_xu_y+uu_{xy})\\
&+v_1^*(v_t+u_xv+uv_x+u_x+u_{xxy})]\\
&-f_2u_y\left(\frac{1}{2}u_1^*u_x-\frac{1}{2}u_{1t}^*-\frac{1}{2}uu_{1x}^*+\frac{1}{3}v_{1xx}^*\right)\\
&-f_2u_{xy}\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)-\frac{1}{3}f_2u_{xxy}v_1^*-\frac{1}{2}f_2u_{yt}u_1^*.
\end{aligned}
$$

Case 3 For the generator

$$X_3=f_3\frac{\partial}{\partial x}+f_{3t}\frac{\partial}{\partial u},$$

the Lie characteristic functions are

$$W^1=f_{3t}-f_3u_x,\ \ W^2=-f_3v_x,$$

we derive the conservation vector of system (1)

$$
\begin{aligned}
C^1 =& -f_3v_xv_1^*-\frac{1}{2}[(f_{3t}-f_3u_x)u_{1y}^*+f_3u_{xy}u_1^*],\\
C^2 =& f_3[u_1^*(u_{yt}+v_{xx}+u_xu_y+uu_{xy})\\
&+v_1^*(v_t+u_xv+uv_x+u_x+u_{xxy})]\\
&+(f_{3t}-f_3u_x)\left(\frac{1}{2}u_1^*u_y+v_1^*v-\frac{1}{2}uu_{1y}^*+\frac{1}{3}v_{1xy}^*\right)\\
&-f_3v_x(v_1^*u-u_{1x}^*)-f_3u_{xy}\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)\\
&+\frac{1}{3}f_3u_{xx}v_{1y}^*-f_3v_{xx}u_1^*-\frac{2}{3}f_3v_1^*u_{xxy},\\
C^3 =& (f_{3t}-f_3u_x)\left[\frac{1}{2}(u_1^*u_x-u_{1t}^*-uu_{1x}^*)+\frac{1}{3}v_{1xx}^*\right]\\
&-f_3u_{xx}\left(\frac{1}{2}uu_1^*-\frac{1}{3}v_{1x}^*\right)\\
&-\frac{1}{3}f_3u_{xxx}v_1^*+\frac{1}{2}(f_{3tt}-f_{3t}u_x-f_3u_{xt})u_1^*.
\end{aligned}
$$

Conservation laws are mathematical expressions of physical laws, such as conservation of energy, mass, and momentum. Not all of the conservation laws of a PDE have physical interpretations but the existence of a great number of conservation laws is a strong indication of its integrability. The celebrated Noether theorem provides us with a corresponding relationship between symmetry and conservation law; she pointed out that each kind of symmetry corresponds to a conservation law, such as the invariance of the spatial transformation ensures the conservation of momentum and the invariance of the temporal transformation guarantees the energy conservation, and vice versa. Therefore, it is of great significance to find infinite conservation laws for a soliton system.

Remark 3 Clearly, the above conservation vectors C^i $(i = 1,2,3)$ include an arbitrary solution u_1^*, v_1^* to adjoint Eq. (37), so the number of conservation laws is infinite.

5. Discussion and summary

With the standard truncated Painlevé expansion, we explore the non-auto Bäcklund transformation, residual symmetry, and interaction solution between soliton-cnoidal periodic waves of the (2+1)-dimensional DLW system (1). Nonlocal in the original nonlinear system, the residual symmetries are localized with a properly auxiliary dependent variable introduced. Besides, solving the standard Lie's initial value problem leads to the finite transformation of the residual symmetry. Moreover, the DLW system is proved to be CRE solvable, and also CTE solvable in a special case. In the CTE, abundant soliton and cnoidal periodic wave solutions can be given by the Jacobi elliptic functions. Furthermore, countless conservation laws of system (1) have been obtained from Ibragimov's new conservation laws. The basic conserved quantity can be applied in obtaining various estimates for smooth solutions and defining suitable norms for weak solutions. So it is worth being further investigated.

A good understanding of the solutions to system (1) is very helpful for coastal and civil engineers by applying the nonlinear water model to coastal harbor design. One can also consider the relationship between residual symmetry and other nonlocal symmetries. The research into the CTE method and more types of interaction solutions to different kinds of nonlinear excitations should be pushed forward.

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