Nonlinear Self-Adjointness, Conservation Laws and Soliton-Cnoidal Wave Interaction Solutions of (2+1)-Dimensional Modified Dispersive Water-Wave System*

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(Received September 5, 2016; revised manuscript received November 14, 2016)

Abstract *This paper mainly discusses the (2+1)-dimensional modified dispersive water-wave (MDWW) system which will be proved nonlinear self-adjointness. This property is applied to construct conservation laws corresponding to the symmetries of the system. Moreover, via the truncated Painlevé analysis and consistent tanh-function expansion (CTE) method, the soliton-cnoidal periodic wave interaction solutions and corresponding images will be eventually achieved.*

PACS numbers: 02.30.Ik, 05.45.Yv, 04.20.Jb **DOI:** 10.1088/0253-6102/67/1/15

Key words: MDWW system, nonlinear self-adjointness, conservation laws, truncated Painlevé analysis, CTE method, interaction solutions

1 Introduction

Conservation laws, essential in the study of differential equations mathematically and physically, propose one of the primary principles to formulate and investigate models, especially in existence, uniqueness and stability of solutions. In addition, the integrability of the system is quite possible should conservation laws exist in it.[1−2] For conservation laws, different methods have been mobilized. The celebrated Noether's theorem[3] proves to be a systematic and efficient approach in finding conservation laws of PDEs unless there exists a Lagrangian. However, there exist some equations not having a Lagrangian. Hence the Noether's theorem cannot be used to obtain conservation laws directly because of the equation symmetries. This, however, can be solved with the general concept of nonlinear self-adjointness proposed by Ibragimov,[4−7] and Gandarias to construct the conservation laws for any differential equation.[8] This procedure can be true of classes of single differential equations of any order but of the systems where the number of equations is equal to that of dependent variables.[9−11]

On the other hand, it is an important and major subject to seek exact solutions and interactions among solutions to nonlinear equation to explain some physical phenomena further. The special solutions to an integrable system can be derived from many effective methods such as symmetry reductions,[12] the variable separation approach,[13] the inverse scattering transformation approach,[14] the Darboux transformation (DT),[15−16] the Bäcklund transformation (BT),[17] the bilinear method,[18] and Painlevé analysis,[19] to name just a few. However, it is difficult to find the interaction solutions among different types of nonlinear excitations besides the soliton-soliton interaction. Recently, Lou *et al.* made a breakthrough in interaction solutions between solitons and any other types of nonlinear soliton waves by using two equivalent simple methods: the truncated Painlevé analysis and the generalized tanh expansion approaches,[20−21] which are proved to be effective for more types of solutions to many integrable systems.

This paper concentrates on investigating the nonlinear self-adjointness, conservation laws and interaction solutions between a soliton and cnoidal wave[22−26] of the (2+1)-dimensional modified dispersive water-wave (MDWW) system, which can be written as

$$\begin{aligned} &F_1 = u_{yt} + u_{xxy} - 2v_{xx} - u_{xy}^2 = 0\,,\\ &F_2 = v_t - v_{xx} - 2u_x v - 2uv_x = 0\,, \end{aligned} \tag{1}$$

system (1), modeling nonlinear and dispersive long gravity waves in two horizontal directions on shallow waters of uniform depth. MDWW is derived from the famous Kadomtsev–Petviashvili (KP) equation with the symmetry constraints.[27] In Refs. [28]–[29], Painlevé–Bäcklund transformations, along with a multilinear variable separation approach help a lot in securing abundant propagating localized excitations. Reference [30] shows many new types of non-traveling solutions acquired via a further generalized projective Riccati equation method. In [31], the extended mapping approach assists in getting some non-propagating and propagating solitons. Reference [32] en-

*Supported by National Natural Science Foundation of China under Grant Nos. 11371293, 11505090, the Natural Science Foundation of Shaanxi Province under Grant No. 2014JM2-1009, Research Award Foundation for Outstanding Young Scientists of Shandong Province under Grant No. BS2015SF009 and the Science and Technology Innovation Foundation of Xi'an under Grant No. CYX1531WL41

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gages in new types of interactions between solitons such as a compacton-like semi-foldon and a compacton, a peakon-like semi-foldon and a peakon based on new variable separation solutions with arbitrary functions for MDWW (1) by using the projective Riccati equation expansion. In Ref. [33], special types of periodic folded waves are derived from the WTC truncation method. In Ref. [34], Hirota bilinear method is of great assistance in constructing multiple soliton solutions with arbitrary functions for system (1). For system (1), Ref. [35] emphasizes symmetry reduction. However, the research into the nonlinear self-adjoint, conservation law and soliton-cnodial wave solution of Eqs. (1) have not been mentioned in the above literature.

This paper is arranged as follows. Section 2 introduces the main notations and theorems used in this paper. In Sec. 3, the nonlinear self-adjointness for the (2+1)-dimensional (MDWW) system will be discussed, which is a vital link in applying Ibragimov's theorem. In Sec. 4, based on Lie symmetry analysis acquired and Ibragimov's theorem, conservation laws of system (1) are established. In Sec. 5, we derive new explicit interactions solutions between solitons and cnoidal periodic waves by the truncated Painlevé analysis and the consistent tanh expansion (CTE) method for the (2+1)-dimensional MDWW system. In the last section, some conclusions and discussion will be given.

2 Preliminaries

This section aims to present the notations and theorems used in this paper.

Definition 1 (Ref. [6]) Consider a system of equations

$$F_\alpha(x,u,u_{(1)},\ldots,u_{(s)})=0,\quad \alpha=1,\ldots m\,,\tag{2}$$

with n independent variables $x=(x^1,\ldots,x^n)$, m dependent variables $u=(u^1,\ldots,u^m)$ and where $u_{(s)}$ denotes the set of the partial derivatives of s-th order of u. The adjoint equation to Eqs. (2) is

$$F^*_\alpha(x,u,v,\ldots,u_{(s)},v_{(s)})=0,\quad \alpha=1,\ldots,m\,,\tag{3}$$

with

$$F^*_\alpha(x,u,v,\ldots,u_{(s)},v_{(s)})=\frac{\delta L}{\delta u^\alpha},\quad \alpha=1,\ldots,m\,,\tag{4}$$

where L is the formal Lagrangian for Eq. (2) given by

$$L=\sum_{\beta=1}^{m}v^\beta F_\beta(x,u,u_{(1)},\ldots,u_{(s)})\,,\tag{5}$$

with $v=(v^1,\ldots,v^m)$ as new dependent variables, $v^\alpha=v^\alpha(x)$, and $\delta/\delta u^\alpha$ as the variational derivative

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,.$$

Definition 2 (Ref. [7]) The system (2) is said to be nonlinearly self-adjoint if the following equations hold:

$$F^*_\alpha(x,u,v,\ldots,u_{(s)},v_{(s)})|_{v^\alpha=\phi^\alpha(x,u)}=\lambda^\beta_\alpha F_\beta(x,u,\ldots,u_{(s)}),\quad \alpha=1,\ldots,m\,,\tag{6}$$

with $\phi(x,u)\neq 0$, where λ^β_α are undetermined coefficients, and ϕ is the m-dimensional vector $\phi=(\phi^1,\ldots,\phi^m)$.

In Ref. [6], Eqs. (3) succeeds the symmetries of the system (2), which has been proved by Ibragimov. In other words, if the system (2) admits a point transformation group with a generator

$$X=\xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u^\alpha}\,,\tag{7}$$

then the adjoint system (3) admits the operator (7) extended to the variables v^α by the formula

$$Y=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\eta^\alpha_*\frac{\partial}{\partial v^\alpha}\,.\tag{8}$$

Theorem 1 (Ref. [6]) Any infinitesimal symmetry (Lie point, Lie Bäcklund, nonlocal)

$$X=\xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u^\alpha}$$

of a system equations (2) provides a conservation law $D_i(C^i)=0$ for the system of differential equations consisting of Eqs. (2) and the adjoint Eqs. (3). The conserved vector is given by

$$\begin{aligned}C^i=\xi^i L&+W^\alpha\Big[\frac{\partial L}{\partial u^\alpha_i}-D_j\Big(\frac{\partial L}{\partial u^\alpha_{ij}}\Big)+D_jD_k\Big(\frac{\partial L}{\partial u^\alpha_{ijk}}\Big)-\cdots\Big]\\&+D_j(W^\alpha)\Big[\frac{\partial L}{\partial u^\alpha_{ij}}-D_k\Big(\frac{\partial L}{\partial u^\alpha_{ijk}}\Big)-\cdots\Big]\\&+D_jD_k(W^\alpha)\Big[\frac{\partial L}{\partial u^\alpha_{ijk}}-\cdots\Big]\,,\end{aligned}\tag{9}$$

and $W^\alpha=\eta^\alpha-\xi^j u^\alpha_j$.

3 Nonlinear Self-Adjointness of System (1)

For system (1), according to Definition 1, the following formal Lagrangian can be deduced

$$\begin{aligned}L&=u_1^*(u_{yt}+u_{xxy}-2v_{xx}-u^2_{xy})\\&\quad+v_1^*(v_t-v_{xx}-2u_xv-2uv_x)\,,\end{aligned}\tag{10}$$

where u_1^* and v_1^* are two new dependent variables. The adjoint system of the system (1) is

$$F_1^*=\frac{\delta L}{\delta u}=0,\quad F_2^*=\frac{\delta L}{\delta v}=0\,,\tag{11}$$

where, in this case

$$\begin{aligned}\frac{\delta L}{\delta u}&=\frac{\partial L}{\partial u}-D_x\frac{\partial L}{\partial u_x}-D_y\frac{\partial L}{\partial u_y}+D_yD_t\frac{\partial L}{\partial u_{yt}}\\&\quad+D_xD_y\frac{\partial L}{\partial u_{xy}}-D_xD_xD_y\frac{\partial L}{\partial u_{xxy}}\,,\\\frac{\delta L}{\delta v}&=\frac{\partial L}{\partial v}-D_x\frac{\partial L}{\partial v_x}-D_t\frac{\partial L}{\partial v_t}+D_xD_x\frac{\partial L}{\partial v_{xx}}\,,\end{aligned}$$

with D_x, D_y and D_t denoting the operator of total differentiation with x, y, and t respectively. Should Eq. (10) be considered, the adjoint system (11) for system (1) will change into

$$\begin{aligned}F_1^*&=u^*_{1yt}+2vv^*_{1x}-2uu^*_{1xy}-u^*_{1xxy}=0\,,\\F_2^*&=v^*_{1t}-2uv^*_{1x}+2u^*_{1xx}+v^*_{1xx}=0\,.\end{aligned}\tag{12}$$

System (1) is not recovered if u is substituted for u_1^* and v for v_1^*, so system (1) is not self adjoint.[10] Based on Definition 2, nonlinearly self-adjoint will the system (1) become if each equation F_i^* $(i=1,2)$ of the adjoint system (12) satisfies the following condition

$$F_1^* = \lambda_{11}F_1 + \lambda_{12}F_2, \quad F_2^* = \lambda_{21}F_1 + \lambda_{22}F_2\,, \tag{13}$$

with regular undetermined coefficients λ_{ij} $(i,j=1,2)$ after substituting the following expression

$$u_1^* = \phi(x,y,t,u,v), \quad v_1^* = \psi(x,y,t,u,v)\,, \tag{14}$$

with $\phi(x,y,t,u,v)\neq 0$ or $\psi(x,y,t,u,v)\neq 0$. Were the differential consequences of (14) to be introduced, system (12) split into the following equations for the coefficients λ_{ij} $(i,j=1,2)$

$$\begin{aligned}&\lambda_{11}=\phi_u, \quad \lambda_{12}=\phi_{yv}\,,\\ &\lambda_{21}=\frac{1}{2}(\phi_v+\psi_v), \quad \lambda_{22}=-\psi_v\,,\end{aligned}$$

and into the system for the substitution (14)

$$\begin{aligned}&\phi_u=\phi_v=\psi_u=\psi_v=\psi_x=0, \quad \phi_{yt}-\phi_{xxy}=0\,,\\ &\phi_{xy}=0, \quad \psi_t+\psi_{xx}+2\phi_{xx}=0\,.\end{aligned} \tag{15}$$

Once they are solved, the following solution will come

$$\phi=\frac{1}{2}\dot g_1x^2+g_2x+g_3+g_4, \quad \psi=-2g_1+g_4\,, \tag{16}$$

where g_1,g_2,g_3 are arbitrary functions of t, and g_4 of y, and the dot over the function denotes its derivative with respect to its variable. Then, according to the Definition 2, system (1) is nonlinearly self adjoint.

4 Lie Symmetries and Conservation Laws of System (1)

The performance of corresponding Lie symmetry analysis by classical lie group method is the prerequisite to derive conservation laws for system (1). It needs to consider a one-parameter Lie group of infinitesimal transformations

$$\begin{aligned}&x\to x+\epsilon\xi^1(x,y,t,u,v), \quad y\to y+\epsilon\xi^2(x,y,t,u,v)\,,\\ &t\to t+\epsilon\xi^3(x,y,t,u,v), \quad u\to u+\epsilon\eta^1(x,y,t,u,v)\,,\\ &v\to v+\epsilon\eta^2(x,y,t,u,v)\,,\end{aligned} \tag{17}$$

with a small parameter $\epsilon\ll 1$. The vector field related to the above transformations can be described as

$$X=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\xi^3\frac{\partial}{\partial t}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v}\,. \tag{18}$$

Then the invariance of system (1) under transformation (17) makes the functions ξ^1,ξ^2,ξ^3, η^1,η^2 take the form

$$\begin{aligned}&\xi^1=\frac{1}{2}\dot f_2x+f_3, \quad \xi^2=f_1, \quad \xi^3=f_2\,,\\ &\eta^1=-\frac{1}{2}\dot f_2u+\frac{1}{2}\ddot f_2x+\dot f_3\,,\\ &\eta^2=-\frac{1}{2}v(2\dot f_1+\dot f_2)\,,\end{aligned} \tag{19}$$

where f_1 is arbitrary function of y, f_2,f_3 of t, and the dot over the functions means their derivative with respect to their variable. An infinite-dimensional Lie algebra of symmetries is resulted from the existence of the arbitrary functions. A general element of this algebra is depicted as

$$X=X_1+X_2+X_3\,, \tag{20}$$

where

$$X_1=f_1\frac{\partial}{\partial y}-\dot f_1v\frac{\partial}{\partial v}\,, \tag{21}$$

$$\begin{aligned}X_2=&\frac{1}{2}\dot f_2\frac{\partial}{\partial x}+f_2\frac{\partial}{\partial t}-\frac{1}{2}\Big(u\dot f_2+\frac{1}{2}x\ddot f_2\Big)\frac{\partial}{\partial u}\\ &-\frac{1}{2}\dot f_2v\frac{\partial}{\partial v}\,,\end{aligned} \tag{22}$$

$$X_3=f_3\frac{\partial}{\partial x}+\dot f_3\frac{\partial}{\partial u}\,. \tag{23}$$

What follows is to apply the Theorem 1 to seek for conservation laws of system (1). For (1), the adjoint equation is given by

$$\begin{aligned}&F_1^*=u_{1yt}^*+2vv_{1x}^*-2uu_{1xy}^*-u_{1xxy}^*=0\,,\\ &F_2^*=v_{1t}^*-2uv_{1x}^*+2u_{1xx}^*+v_{1xx}^*=0\,,\end{aligned} \tag{24}$$

and the Lagrangian in the symmetrized form

$$L=u_1^*\Big(\frac{1}{2}u_{yt}+\frac{1}{2}u_{ty}-2u_xu_y-uu_{xy}-uu_{yx}+\frac{1}{3}u_{xxy}+\frac{1}{3}u_{xyx}+\frac{1}{3}u_{yxx}-2v_{xx}\Big)+v_1^*(v_t-2u_xv-2uv_x-v_{xx})\,. \tag{25}$$

Consider Theorem 1, the corresponding vector fields can be written as

$$X=\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\xi^3\frac{\partial}{\partial y}+\eta^1\frac{\partial}{\partial u}++\eta^2\frac{\partial}{\partial v}\,. \tag{26}$$

The conservation law is decided by

$$D_t(C^1)+D_x(C^2)+D_y(C^3)=0\,. \tag{27}$$

Here the conserved vector $C=(C^1,C^2,C^3)$ is given by (9) and the concrete forms are as follows

$$\begin{aligned}C^1=&\xi^1L+W^2\frac{\partial L}{\partial v_t}-W^1D_y\frac{\partial L}{\partial u_{ty}}+W_y^1\frac{\partial L}{\partial u_{ty}}\,,\\ C^2=&\xi^2L+W^1\Big(\frac{\partial L}{\partial u_x}-D_y\frac{\partial L}{\partial u_{xy}}+D_{xy}\frac{\partial L}{\partial u_{xxy}}+D_{yx}\frac{\partial L}{\partial u_{xyx}}\Big)+W^2\Big(\frac{\partial L}{\partial v_x}-D_x\frac{\partial L}{\partial v_{xx}}\Big)\\ &+W_y^1\Big(\frac{\partial L}{\partial u_{xy}}-D_x\frac{\partial L}{\partial u_{xyx}}\Big)-W_x^1D_y\frac{\partial L}{\partial u_{xxy}}+W_x^2\frac{\partial L}{\partial v_{xx}}+W_{xy}^1\frac{\partial L}{\partial u_{xxy}}+W_{yx}^1\frac{\partial L}{\partial u_{xyx}}\,,\end{aligned}$$

$$C^3 = \xi^3 L + W^1\Big(\frac{\partial L}{\partial u_y} - D_x\frac{\partial L}{\partial u_{yx}} + D_{xx}\frac{\partial L}{\partial u_{yxx}}\Big) + W^1_t\frac{\partial L}{\partial u_{yt}} + W^1_x\Big(\frac{\partial L}{\partial u_{yx}} - D_x\frac{\partial L}{\partial u_{yxx}}\Big) + W^1_{xx}\frac{\partial L}{\partial u_{yxx}}\,. \tag{28}$$

Substituting (25) into (28), it will change into

$$C^1 = \xi^1 L + W^2 v_1^* - \frac{1}{2}(W^1 u_{1y}^* - u_1^* W_y^1)\,,$$

$$C^2 = \xi^2 L + W^1\Big(-2u_1^* u_y - 2v_1^* v + u_y u_1^* + u u_{1y}^* + \frac{2}{3}u_{1xy}^*\Big) + W^2(-2uv_1^* + 2u_{1x}^* + v_{1x}^*)$$
$$- W_y^1\Big(uu_1^* + \frac{1}{3}u_{1x}^*\Big) - \frac{1}{3}W_x^1 u_{1y}^* - W_x^2(2u_1^* + v_1^*) + \frac{2}{3}u_1^* W_{xy}^1\,,$$

$$C^3 = \xi^3 L + W^1\Big(\frac{1}{3}u_{1xx}^* - 2u_1^* u_x - \frac{1}{2}u_{1t}^* + u_x u_1^* + u u_{1x}^*\Big) + \frac{1}{2}W_t^1 u_1^* - W_x^1\Big(uu_1^* + \frac{1}{3}u_{1x}^*\Big) + \frac{1}{3}u_1^* W_{xx}^1\,,$$

with

$$W^1 = \eta^1 - \xi^1 u_t - \xi^2 u_x - \xi^3 u_y, \quad W^2 = \eta^2 - \xi^1 v_t - \xi^2 v_x - \xi^3 v_y\,.$$

In regard to (21), we consider the following cases.

Case 1 For the generator

$$X_1 = f_1\frac{\partial}{\partial y} - v f_{1y}\frac{\partial}{\partial v}\,,$$

the Lie characteristic functions are

$$W^1 = -f_1 u_y, \quad W^2 = -v f_{1y} - f_1 v_y\,,$$

one can obtain the conservation vector of (1)

$$C^1 = -(vf_{1y} + f_1 v_y)v_1^* + \frac{1}{2}(f_1 u_y u_{1y}^* - u_1^* f_{1y} u_y - u_1^* f_1 u_{yy})\,,$$

$$C^2 = f_1 u_y\Big(2u_1^* u_y + 2v_1^* v - u_y u_1^* - u u_{1y}^* - \frac{2}{3}u_{1xy}^*\Big) - (vf_{1y} + f_1 v_y)(v_{1x}^* - 2uv_1^* + 2u_{1x}^*)$$
$$+ (f_{1y}u_y + f_1 u_{yy})\Big(uu_1^* + \frac{1}{3}u_{1x}^*\Big) + \frac{1}{3}u_{1y}^* f_1 u_{yx} + (v_x f_{1y} + f_1 v_{yx})(2u_1^* + v_1^*) - \frac{2}{3}u_1^*(f_{1y}u_{xy} + f_1 u_{xyy})\,,$$

$$C^3 = f_1 u_1^*(u_{yt} - 2u_x u_y - 2uu_{xy} + u_{xxy} - 2v_{xx}) + f_1 v_1^*(v_t - 2u_x v - 2uv_x - v_{xx})$$
$$- f_1 u_y\Big(\frac{1}{3}u_{1xx}^* - 2u_1^* u_x - \frac{1}{2}u_{1t}^* + u_x u_1^* + uu_{1x}^*\Big) + f_1 u_{yx}\Big(uu_1^* + \frac{1}{3}u_{1x}^*\Big) - \frac{1}{2}u_1^* f_1 u_{yt} - \frac{1}{3}u_1^* f_1 u_{yxx}\,.$$

Case 2 For the generator

$$X_2 = \frac{1}{2}xf_{2t}\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial t} - \frac{1}{2}\Big(uf_{2t} + \frac{1}{4}xf_{2tt}\Big)\frac{\partial}{\partial u} - \frac{1}{2}vf_{2t}\frac{\partial}{\partial v}\,,$$

the Lie characteristic functions are

$$W^1 = -\frac{1}{2}\Big(uf_{2t} + \frac{1}{2}xf_{2tt}\Big) - f_2 u_t - \frac{1}{2}xf_{2t}u_x, \quad W^2 = -\frac{1}{2}vf_{2t} - f_2 v_t - \frac{1}{2}xf_{2t}v_x\,,$$

we can get the conservation vector of (1)

$$C^1 = f_2 u_1^*(u_{ty} - 2u_x u_y - 2uu_{xy} + u_{xxy} - 2v_{xx}) + f_2 v_1^*(v_t - 2u_x v - 2uv_x - v_{xx})$$
$$- \frac{1}{2}v_1^*\Big(vf_{2t} + f_2 v_t + \frac{1}{2}xf_{2t}v_x\Big) + \Big[\frac{1}{4}\Big(uf_{2t} + \frac{1}{2}xf_{2tt}\Big) + f_2 u_t + \frac{1}{2}xf_{2t}u_x\Big]u_{1y}^*$$
$$- \frac{1}{2}u_1^*\Big(\frac{1}{2}u_y f_{2t} + f_2 u_{ty} + \frac{1}{2}xf_{2t}u_{xy}\Big)\,,$$

$$C^2 = \frac{1}{2}xf_{2t}u_1^*(u_{yt} - 2u_x u_y - 2uu_{xy} + u_{xxy} - 2v_{xx}) + \frac{1}{2}xf_{2t}v_1^*(v_t - 2u_x v - 2uv_x - v_{xx})$$
$$+ \Big(\frac{1}{2}uf_{2t} + \frac{1}{8}xf_{2tt} + f_2 u_t + \frac{1}{2}xf_{2t}u_x\Big)\Big(2u_1^* u_y + 2v_1^* v - u_y u_1^* - uu_{1y}^* - \frac{2}{3}u_{1xy}^*\Big)$$
$$- \Big(\frac{1}{2}vf_{2t} + f_2 v_t + \frac{1}{2}xf_{2t}v_x\Big)(v_{1x}^* - 2uv_1^* + 2u_{1x}^*) + \Big(\frac{1}{2}u_y f_{2t} + f_2 u_{ty} + \frac{1}{2}xf_{2t}u_{xy}\Big)\Big(uu_1^* + \frac{1}{3}u_{1x}^*\Big)$$
$$+ \frac{1}{3}u_{1y}^*\Big(\frac{1}{2}u_x f_{2t} + \frac{1}{4}f_{2tt} + f_2 u_{tx} + \frac{1}{2}f_{2t}u_x + \frac{1}{2}xf_{2t}u_{xx}\Big)$$
$$+ \Big(\frac{1}{2}v_x f_{2t} + f_2 v_{tx} + \frac{1}{2}f_{2t}v_x + \frac{1}{2}xf_{2t}v_{xx}\Big)(2u_1^* + v_1^*) - \frac{1}{3}u_1^*(2f_{2t}u_{xy} + 2f_2 u_{txy} + f_{2t}xu_{xxy})\,,$$

$$C^3 = -\Big(\frac{1}{2}uf_{2t} + \frac{1}{4}xf_{2tt} + f_2 u_t + \frac{1}{2}xf_{2t}u_x\Big)\Big(\frac{1}{3}u_{1xx}^* - 2u_1^* u_x - \frac{1}{2}u_{1t}^* + u_x u_1^* + uu_{1x}^*\Big)$$

$$+\Big(\frac{1}{2}u_x f_{2t}+\frac{1}{4}f_{2tt}+f_2u_{tx}+\frac{1}{2}f_{2t}u_x+\frac{1}{2}xf_{2t}u_{xx}\Big)\Big(uu_1^*+\frac{1}{3}u_{1x}^*\Big)$$
$$-\frac{1}{2}u_1^*\Big(\frac{1}{2}u_t f_{2t}+\frac{1}{2}uf_{2tt}+\frac{1}{4}xf_{2ttt}+f_{2t}u_t+f_2u_{tt}+\frac{1}{2}xf_{2tt}u_x+\frac{1}{2}xf_{2t}u_{xt}\Big)$$
$$-\frac{1}{6}u_1^*(3f_{2t}u_{xx}+2f_2u_{txx}+f_{2t}xu_{xxx})\,.$$

Case 3 For the generator

$$X_3=f_3\frac{\partial}{\partial x}+f_{3t}\frac{\partial}{\partial u}\,,$$

the Lie characteristic functions are

$$W^1=f_{3t}-f_3u_x,\quad W^2=-f_3v_x\,,$$

we derive the conservation vector of (1)

$$C^1=-f_3v_xv_1^*-\frac{1}{2}[(f_{3t}-f_3u_x)u_{1y}^*+f_3u_{xy}u_1^*]\,,$$
$$C^2=f_3u_1^*(u_{yt}-2u_xu_y-2uu_{xy}+u_{xxy}-2v_{xx})+f_3v_1^*(v_t-2u_xv-2uv_x-v_{xx})$$
$$+(f_{3t}-f_3u_x)\Big(\frac{2}{3}u_{1xy}^*-2u_1^*u_y-2v_1^*v+u_yu_1^*+uu_{1y}^*\Big)-f_3v_x(2u_{1x}^*-2uv_1^*$$
$$+v_{1x}^*)+f_3u_{xy}\Big(uu_1^*+\frac{1}{3}u_{1x}^*\Big)+\frac{1}{3}f_3u_{xx}u_{1y}^*+f_3v_{xx}(2u_1^*+v_1^*)-\frac{2}{3}f_3u_1^*u_{xxy}\,,$$
$$C^3=(f_{3t}-f_3u_x)\Big(\frac{1}{3}u_{1xx}^*-2u_1^*u_x-\frac{1}{2}u_{1t}^*+u_xu_1^*+uu_{1x}^*\Big)+f_3u_{xx}\Big(uu_1^*+\frac{1}{3}u_{1x}^*\Big)$$
$$+\frac{1}{2}u_1^*(f_{3tt}-f_{3t}u_x-f_3u_{xt})-\frac{1}{3}u_1^*f_3u_{xx}\,.$$

Remark 1 Clearly, the above conservation vector C^i $(i=1,2,3)$ includes an arbitrary solution u_1^*, v_1^* to adjoint Eqs. (24), so the number of the conservation laws it presents is infinite.

5 Soliton-Cnoidal Wave Interaction Solutions of System (1)

Obviously, the Painlevé analysis is one of the effective approaches for special solutions to nonlinear physical systems. For the (2+1)-dimensional MDWW system, its truncated Painlevé expansion can be expressed as

$$u=\frac{u_1}{\phi}+u_0,\quad v=\frac{v_2}{\phi^2}+\frac{v_1}{\phi}+v_0\,,\tag{29}$$

with u_0,u_1,v_0,v_1,v_2,ϕ being the functions of x,y and t. By substituting Eq. (29) into system (1) and vanishing all the coefficients of different powers of $1/\phi$ comes

$$u_0=\phi_x,\quad u_1=2\phi_x,\quad u_1=\frac{\phi_t-\phi_{xx}}{2\phi_x}\,,$$
$$v_0=-\phi_y\phi_x,\quad v_1=\phi_{xy}\,,$$
$$v_2=\frac{-1}{2\phi_x^2}(\phi_{xy}\phi_t-\phi_{yt}\phi_x-\phi_{xy}\phi_{xx}+\phi_{xxy}\phi_x)\,,\tag{30}$$

and then we obtain

$$u=\frac{2\phi_x^2+\phi\phi_t-\phi\phi_{xx}}{2\phi_x\phi}\,,$$
$$v=\frac{1}{-2\phi_x^2\phi^2}(2\phi_y\phi_x^3-2\phi_{xy}\phi\phi_x^2+\phi_{xy}\phi^2\phi_t$$
$$-\phi_{xy}\phi^2\phi_{xx}-\phi_{yt}\phi_x\phi^2+\phi_{xxy}\phi_x\phi^2)\,,\tag{31}$$

which is the solution to the MDWW system, and the field ϕ satisfies the following Schwarzian form

$$\frac{-1}{2}CC_x+\frac{1}{2}S_x+\frac{1}{2}C_t-C_{xx}+\lambda=0\,,\tag{32}$$

where λ is an arbitrary integral parameter, and

$$C=\frac{\phi_t}{\phi_x},\quad S=\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2.$$

The Schwarzian form (32) is invariant under the Möbious transformation

$$\phi\to\frac{a+b\phi}{c+d\phi}\quad(ad\neq bc)\,.$$

That is to say, Eq. (32) bears three symmetries $\sigma^\phi=d_1$, $\sigma^\phi=d_2\phi$, and $\sigma^\phi=d_3\phi^2$ with arbitrary constants d_1, d_2 and d_3.

Adopting the following straightening transformation,

$$\phi=\frac{2}{\tanh(w)-1}\,,\tag{33}$$

where w is the function of x, y, and t. After substituting the expression (33) into system (31), the equivalent solutions to MDWW system come as

$$u=w_x\tanh(w)-\frac{w_{xx}-w_t}{2w_x}\,,$$
$$v=-w_xw_y\tanh^2(w)+w_{xy}\tanh(w)+w_xw_y$$
$$+\frac{w_{yt}-w_{xxy}}{2w_x}+\frac{w_{xx}w_{xy}-w_tw_{xy}}{2w_x^2}\,,\tag{34}$$

and the equivalent compatibility condition for w as

$$\frac{1}{2}C_{1t}-\frac{1}{2}C_1C_{1x}-C_{1xx}+\frac{1}{2}S_{1x}+2w_xw_{xx}=0\,,\tag{35}$$

where

$$C_1 = \frac{w_t}{w_x}, \quad S_1 = \frac{w_{xxx}}{w_x} - \frac{3}{2}\Big(\frac{w_{xx}}{w_x}\Big)^2 .$$

Clearly, the solutions (34) are derived from the transformation (33), where the usual truncated Painlevé expansion approach is converted into the most general extension of the special tanh function expansion method, so it can be said the solutions (34) are the generalization of the usual tanh function expansion method. Here we can obtain the solution (34) by the CTE approaches.[36]

For the MDWW system (1), the application of leading order analysis can result in the following generalized truncated tanh function expansion

$$u = u_0 + u_1 \tanh(w),$$

$$v = v_0 + v_1 \tanh(w) + v_2 \tanh^2(w), \tag{36}$$

where u_0, u_1, v_0, v_1, v_2 and w are functions of x, y, and t. Substituting expression (36) into system (1) and vanishing all the coefficients of $\tanh^i(w)$, we have

$$\begin{aligned} u_0 &= \frac{w_t - w_{xx}}{2w_x}, \quad u_1 = w_x, \\ v_1 &= w_{xy}, \quad v_2 = -w_x w_y, \\ v_0 &= \frac{-1}{2w_x^2}(-2w_y w_x^3 + w_t w_{xy} - w_x w_{yt} + w_x w_{xxy} \\ &\quad - w_{xx} w_{xy}), \end{aligned} \tag{37}$$

and then we deduce the same solution (34) to the MDWW system (1) with the consistent condition (35).

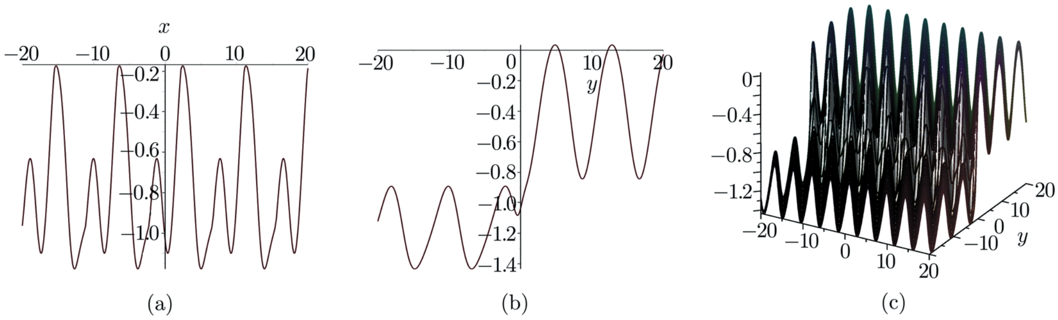

Fig. 1 The soliton-cnodial periodic wave solution to u: (a) The profile of the special structure with $t = 0$ and $y = 0$. (b) The profile of the special structure at $t = 0$ and $x = 0$. (c) Perspective view of the wave.

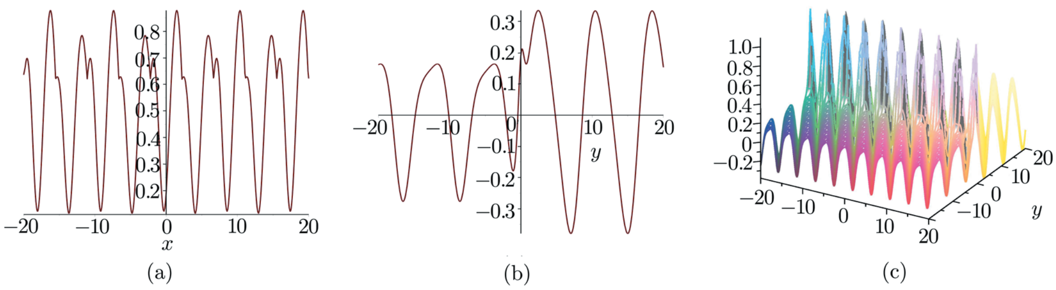

Fig. 2 The soliton-cnodial periodic wave solutions to v: (a) The profile of the special structure with $t = 0$ and $y = 0$. (b) The profile of the special structure with $t = 0$ and $x = 0$. (c) Perspective view of the wave.

The above shows that the single soliton (or solitary wave) solution to the MDWW system (1) is only a straightened solution $w = k_1 x + l_1 y + d_1 t$ to Eq. (35), which implies that to find the interaction solutions between solitons and other nonlinear excitations, what is needed is to acquire the solution to Eq. (35). In this paper we focus on the following special Jacobi elliptic function

$$\begin{aligned} w &= h_1 x + h_2 y + h_3 t \\ &\quad + \lambda E_\pi(\mathrm{sn}(q_1 x + q_2 y + q_3 t, m), n, m), \end{aligned} \tag{38}$$

as the solution to Eq. (35), which characterizes the interactions between a soliton and a cnoidal wave. $h_1, h_2, h_3, q_1, q_2, q_3, \lambda, m$ and n are determined later. In (38), $\mathrm{sn}(z, m)$ is the usual Jacobi elliptic sine function and

$$E_\pi(\zeta, n, m) = \int_0^\zeta \frac{\mathrm{d}t}{(1 - nt^2)\sqrt{(1 - t^2)(1 - m^2 t^2)}}$$

is the third type of incomplete elliptic integral. By substituting (38) into (35) and solving the over-determined equations with the help of *maple* will come

$$\begin{aligned} &h_1 = 0, \quad h_2 = h_2, \quad h_3 = h_3, \quad \lambda = \lambda, \quad m = m, \\ &n = n, \quad q_1 = q_1, \quad q_2 = q_2, \quad q_3 = q_3, \end{aligned} \tag{39}$$

where $h_2, h_3, \lambda, m, n, q_1, q_2$ and q_3 are arbitrary constants. Substituting Eqs. (37), (38), and (39) into (36), we can obtain the interaction solution between soliton and cnoidal periodic waves. The result is omitted here because of its prolixity. Corresponding images are as follows and the parameters used in the figure are selected as $\{h_2 = 1.4, h_3 = -0.5, \lambda = -0.3, q_1 = -0.9, q_2 = -0.5, q_3 = 0.2, m = 0.8, n = 0.5\}$.

Remark 3 Figures 1 and 2 illustrate the soliton-cnoidal periodic wave solutions to the fields u and v describing a soliton travels on a cnoidal wave background for the MDWW system. Clearly, the interaction between the soliton and every peak of the cnoidal periodic wave is elastic as phase changes. Solutions and figures obtained in this paper might be helpful in further understanding the propagation of nonlinear and dispersive long gravity waves on shallow waters.

6 Summary and Discussion

It is proved that the (2+1)-dimensional MDWW system (1) is nonlinearly self-adjoint. With the support of the general theorem of conservation laws by Ibragimov,[6] the property can be applied to construct countless conservation laws for (1). Mathematically, the basic conserved quantity can be applied in obtaining various estimates for smooth solutions and defining suitable norms for weak solutions, so it is worthy to be further investigated.

In addition, with the truncated Painlevé analysis and the CTE method, the soliton-cnoidal wave solution to system (1) is obtained. A good understanding of the solutions to system (1) is very helpful for coastal and civil engineers in applying the nonlinear water model to coastal harbor design. For their practicability, the study on the CTE method and more types of the interaction solutions among different kinds of nonlinear excitations should be furthered.

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